A REMARK ON COEFFICIENTS OF JACOBI MATRICES ARISING FROM A SCHRÖDINGER OPERATOR

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Abstract. A discrete analogue of a Schrödinger type operator proposed by J. Bellissard [6] has a singular continuous spectrum. In this remark we answer the conjecture formulated in [1] on the coefficients of that operator. It turns out that the coefficients have a more complicated behavior than it was conjectured.

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1. Introduction

1.1. **Dyson's example.** Consider the following discrete analog of the Schrödinger operator:

$$(Hu)(n) = p(n)u(n+1) + q(n)u(n-1) + V(n)u(n).$$

acting in the spaces $l_2(Z)$ or $l_2(N)$.

For illustrative purposes we describe the example of F. Dyson (see [2]) where such an operator naturally arises.

Consider a chain of N masses, each coupled to its nearest neighbors by elastic springs obeying Hooke's law. Let all motions take place in one dimension so that each mass is described by a single coordinate.

Let particle number j in the chain have mass m_j, and let its displacement from its equilibrium position be x_j. Let the elastic modules of the spring between particles j and $j+1$ be K_j. Then the equations of the system's motion are:

$$m_j \frac{d^2 x_j}{dt^2} = K_j(x_{j+1} - x_j) + K_{j-1}(x_{j-1} - x_j), \quad j = 2, 3, \dots, N-1.$$

On the particles x_1, x_N situated on the ends we put certain boundary conditions.

It is convenient to introduce new variables:

$$y_j = x_j \sqrt{m_j}$$

and new constants $\{\lambda_j\}_{j=1}^{2N-2}$ given by:

$$\lambda_{2j-1} = \frac{K_j}{m_j}, \quad \lambda_{2j} = \frac{K_j}{m_{j+1}}.$$

Then the equations of motion take the symmetric form:

$$\frac{d^2y_j}{dt^2} = y_{j+1}\sqrt{\lambda_{2j-1}\lambda_{2j}} + y_{j-1}\sqrt{\lambda_{2j-3}\lambda_{2j-2}} - (\lambda_{2j-1}+\lambda_{2j-2})y_j, \quad j = 2,3,\dots,N-1.$$

For each mode

$$y_j(t) = u(j)e^{iEt}$$

we have

$$u(j+1)\sqrt{\lambda_{2j-1}\lambda_{2j}} + u(j-1)\sqrt{\lambda_{2j-3}\lambda_{2j-2}} - (\lambda_{2j-1}+\lambda_{2j-2})u(j) = -E^2u(j),$$

where $j = 2,3,\dots,N-1$.

For physical applications it is important to understand the properties of the spectrum, i.e. the properties of values of E for which the equations have a nontrivial solution.

In relation with this problem, mathematically it is more convenient to investigate the infinite dimensional case.

1.2. **Schrödinger operators.** Consider the following two types of operators:

$$T_1(u)(j) = u(j+1)\sqrt{\lambda_{2j-1}\lambda_{2j}} + u(j-1)\sqrt{\lambda_{2j-3}\lambda_{2j-2}} - (\lambda_{2j-1}+\lambda_{2j-2})u(j), \quad j \in Z \tag{1.1}$$

where $u \in l_2(Z)$; and

$$T_2(u)(j) = u(j+1)\sqrt{\lambda_{2j-1}\lambda_{2j}} + u(j-1)\sqrt{\lambda_{2j-3}\lambda_{2j-2}} - (\lambda_{2j-1}+\lambda_{2j-2})u(j), \quad j \in N \tag{1.2}$$

where $u \in l_2(N)$.

Note that for the first type (1.1) there is no boundary condition while for the second type (1.2) we take into account the existence of the boundary, too.

A typical equation of the first type is the almost Mathieu equation:

$$u(j+1) + u(j-1) + \lambda u(j)\cos 2\pi(\theta - \alpha j) = Eu(j), \quad j \in Z \tag{1.3}$$

where $u \in l_2(Z)$ and α is an irrational number.

A. Avron and S. Jitomirskaya (see [8]) proved:

Theorem 1.4. *Let α be irrational number and $\lambda \neq 0$. Then for almost all θ the spectrum of the operator*

$$T(u)(j) = u(j+1) + u(j-1) + \lambda\cos 2\pi\,(\theta - j\alpha)\,u(j), \quad j \in Z$$

is singular continuous.

A typical equation of the second type is:

$$(Hu)(j) = u(j+1) + u(j-1) + V(j)u(j), \quad j \in N \tag{1.5}$$

where $u \in l_2(N)$.

Following [9, p. 212] let's point out that general wisdom used to say that Schrödinger operators should have absolutely continuous spectrum plus some discrete point spectrum, while singular continuous spectrum is a pathology that should not occur in examples with

V bounded. This general picture was proven to be wrong by Pearson (see [4], [5]) who constructed a potential V such that the operator $H = H_0 + V$ has singular continuous spectrum. His potential V consists of bumps further and further apart with the height of bumps possibly decreasing. Furthermore, singular continuous spectrum occurs in the innocent - looking almost Mathieu equation (1.3).

1.3. **Bellissard's example.** One would like to find a potential function whose properties resemble physical phenomena closer. For example, it would be more preferable to have an almost periodic potential.

With this aim let us consider an operator of the form (1.2) proposed in [6]:

$$(Hu)(j) = \sqrt{R_{j+1}}u(j+1) + \sqrt{R_j}u(j-1), \quad j \in N$$

where the numbers R_n are defined recursively by (2.1),(2.2),(2.3).

In paper [1, p. 134] the following results are proved for $\lambda > 2$:

Proposition 1.

$$0 < R_{2n} < R_n, \quad 0 < R_{2n} \leq 1,$$

Proposition 2.

$$\lim_{k \to \infty} R_{p2^k} = 0,$$

Proposition 3.

$$\lim_{k \to \infty} R_{p2^k+s} = R_s.$$

Let us note that if in Proposition 3 the limit is uniform for p and s then R_n is a limit - periodic sequence (see [10]).

In ([1], p. 135) it was conjectured that the sequence R_n splits in the following way: R_{k2^r+n} lies between R_n and R_{2^r+n}. The authors further point out that would their conjecture be true then the set $\{R_n\}$ would be perfect.

In this paper we prove that for $\lambda > 2$ the sequence R_n splits into 4 parts (see (2.9)). However, there is no further splitting (see (2.15)).

This shows that the structure of the set R_n is much more complicated than it was conjectured in [1].

2. Formulation of the Results

Let $\lambda > 2$. Corresponding to λ let us discuss the numerical sequence R_n (where $n = 0, 1, \ldots$) defined recursively by:

$$R_0 = 0 \tag{2.1}$$

$$R_{2n} + R_{2n+1} = \lambda \tag{2.2}$$

$$R_{2n}R_{2n-1} = R_n \tag{2.3}$$

According to [1, p. 135] numerical studies show that the set $\{R_n\}_{n=0}^{\infty}$ separates into disjoint subsets as follows:

Conjecture 2.4. *Let $\lambda > 2$ and R_n be defined by* (2.1),(2.2),(2.3) *then for any $n = 0, 1, 2, \dots$ the following inequalities hold:*

$$R_0 \le R_{4n} \le R_4, \tag{2.5}$$

$$R_6 \le R_{4n+2} \le R_2, \tag{2.6}$$

$$R_3 \le R_{4n+3} \le R_7, \tag{2.7}$$

and

$$R_5 \le R_{4n+1} \le R_1 \tag{2.8}$$

In the light of this conjecture we prove the following theorem:

Theorem 2.9. *Let $\lambda > 2$ and R_n be defined by* (2.1),(2.2),(2.3) *then for any $n = 0, 1, 2, \dots$ the following inequalities hold:*

$$0 < R_{4n} \le \frac{1}{\lambda - 1} \tag{2.10}$$

$$1 - \frac{1}{\lambda - 1} \le R_{4n+2} < 1 \tag{2.11}$$

$$\lambda - 1 < R_{4n+3} \le \lambda - 1 + \frac{1}{\lambda - 1} \tag{2.12}$$

$$\lambda - \frac{1}{\lambda - 1} \le R_{4n+1} < \lambda \tag{2.13}$$

Remark 2.14. Theorem (2.9) compares with the conjecture (2.4) in the following way:
1. the bounds in the inequalities (2.5) and (2.10) are the same and obviously are sharp;
2. the bounds in the inequalities (2.8) and (2.13) are the same and obviously are sharp;
3. the inequality (2.12) proved in (2.9) is sharper than the inequality (2.7) conjectured in (2.4), indeed one can calculate that for $\lambda > 2$ we have:

$$R_7 = \frac{\lambda^3 - 2\lambda^2 + \lambda - 1}{\lambda^2 - \lambda - 1} > \lambda - 1 + \frac{1}{\lambda - 1}$$

4. the inequality (2.11) proved in theorem (2.9) is weaker than the inequality (2.6) conjectured in (2.4), indeed one can calculate that for $\lambda > 2$ we have:

$$R_6 = \frac{(\lambda - 1)^2}{\lambda^2 - \lambda - 1} > 1 - \frac{1}{\lambda - 1}$$

But as it turns out the conjectured inequality (2.6) is not true, indeed for $\lambda = 2.1$ we have $R_{10} < R_6$. Interestingly, for large values of λ the lower bound of (2.6) seems to be true.

Remark 2.15. In [1, p. 135] it is further conjectured that R_{k2^r+n} lies between R_n and R_{2^r+n}. The same example with $\lambda = 2.1$ and $R_{10} < R_6 < R_2$ comes to prove that this conjecture is not true.

3. Proof of Theorem (2.9)

3.1. **Step 1.** From the recurrent formula (2.2) it follows that:

$$\begin{aligned} R_{4n} + R_{4n+1} &= \lambda \\ R_{4n+2} + R_{4n+3} &= \lambda \end{aligned} \tag{3.1}$$

Also, from the recurrent formula (2.3) it follows that:

$$\begin{aligned} R_{8n}R_{8n-1} &= R_{4n} \\ R_{8n+2}R_{8n+1} &= R_{4n+1} \\ R_{8n+4}R_{8n+3} &= R_{4n+2} \\ R_{8n+6}R_{8n+5} &= R_{4n+3} \end{aligned} \tag{3.2}$$

By combining (3.1) with (3.2) we get:

$$R_{8n}(\lambda - R_{8n-2}) = R_{4n}$$

$$R_{8n+2}(\lambda - R_{8n}) = \lambda - R_{4n}$$

$$R_{8n+4}(\lambda - R_{8n+2}) = R_{4n+2}$$

$$R_{8n+6}(\lambda - R_{8n+4}) = \lambda - R_{4n+2}$$

These can be transformed into the following:

$$\begin{aligned} R_{8n} &= \frac{R_{4n}}{\lambda - R_{8n-2}} \\ 1 - R_{8n+2} = 1 - \frac{\lambda - R_{4n}}{\lambda - R_{8n}} = 1 - \frac{\lambda - R_{4n}}{\lambda - \frac{R_{4n}}{\lambda - R_{8n-2}}} &= \frac{R_{4n}(\lambda - 1 - R_{8n-2})}{\lambda(\lambda - R_{8n-2}) - R_{4n}} \\ R_{8n+4} &= \frac{R_{4n+2}}{\lambda - R_{8n+2}} \\ 1 - R_{8n+6} &= \frac{R_{4n+2}(\lambda - 1 - R_{8n+2})}{\lambda(\lambda - R_{8n+2}) - R_{4n+2}} \end{aligned} \tag{3.3}$$

3.2. **Step 2.** For convenience let's denote:

$$\sigma = \frac{1}{\lambda - 1}$$

Then we can rewrite the conclusion of theorem (2.9) in a concise form as follows:

$$\begin{aligned} 0 < R_{4j} &\leq \sigma \\ 0 < 1 - R_{4j+2} &\leq \sigma \\ \lambda - 1 < R_{4j+3} &\leq \lambda - 1 + \sigma \\ \lambda - \sigma \leq R_{4j+1} &< \lambda \end{aligned} \tag{3.4}$$

We will be proving that system of inequalities by induction over j. Indeed, assume that (3.4) holds for $j < 2n$. We need to prove that

$$\begin{aligned} 0 < R_{4\cdot 2n} &\leq \sigma \\ 0 < 1 - R_{4\cdot 2n+2} &\leq \sigma \\ 0 < R_{4\cdot 2n+4} &\leq \sigma \\ 0 < 1 - R_{4\cdot 2n+6} &\leq \sigma \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \lambda - \sigma \leq R_{4\cdot 2n+1} &< \lambda \\ \lambda - 1 < R_{4\cdot 2n+3} &\leq \lambda - 1 + \sigma \\ \lambda - \sigma \leq R_{4\cdot 2n+5} &< \lambda \\ \lambda - 1 < R_{4\cdot 2n+7} &\leq \lambda - 1 + \sigma \end{aligned} \tag{3.6}$$

In fact, we are only concerned with proving (3.5) as (3.6) will then follow automatically from (2.2).

3.3. **Step 3.** Applying the inequalities (3.3) and the inductive assumption (3.4) we get:

$$\begin{aligned} R_{8n} &= \frac{R_{4n}}{\lambda - R_{8n-2}} \leq \frac{\sigma}{\lambda - 1} \leq \sigma \\ R_{8n} &= \frac{R_{4n}}{\lambda - R_{8n-2}} > 0 \\ 1 - R_{8n+2} &= \frac{R_{4n}(\lambda - 1 - R_{8n-2})}{\lambda(\lambda - R_{8n-2}) - R_{4n}} \geq \frac{\sigma(\lambda - 2)}{\lambda(\lambda - 1 + \sigma)} > 0 \\ 1 - R_{8n+2} &= \frac{R_{4n}(\lambda - 1 - R_{8n-2})}{\lambda(\lambda - R_{8n-2}) - R_{4n}} \leq \frac{\sigma(\lambda - 2 + \sigma)}{\lambda(\lambda - 1) - \sigma} \leq \sigma \end{aligned} \tag{3.7}$$

The very last inequality in (3.7) follows from the following observation:

$$\frac{\sigma(\lambda - 2 + \sigma)}{\lambda(\lambda - 1) - \sigma} \leq \sigma \Leftrightarrow \frac{\lambda - 2 + \sigma}{\lambda(\lambda - 1) - \sigma} \leq 1 \Leftrightarrow \lambda - 2 + 2\sigma \leq \lambda(\lambda - 1) \Leftrightarrow \lambda - 2 + \frac{2}{\lambda - 1} \leq \lambda(\lambda - 1) \Leftrightarrow$$

$$\Leftrightarrow 2 \leq (\lambda - 1)(\lambda^2 - 2\lambda + 2) \Leftrightarrow 2 = min_{2\leq\lambda}(\lambda - 1)((\lambda - 1)^2 + 1)$$

Thus (3.7) proves the first two inequalities of (3.5). As for the other two inequalities of (3.7) we apply the inequalities (3.3) and the inductive assumption (3.4) to get:

$$\begin{aligned} R_{8n+4} &= \frac{R_{4n+2}}{\lambda - R_{8n+2}} \leq \frac{1}{\lambda - R_{8n+2}} \\ 1 - R_{8n+6} &= \frac{R_{4n+2}(\lambda - 1 - R_{8n+2})}{\lambda(\lambda - R_{8n+2}) - R_{4n+2}} \leq \frac{\lambda - 1 - R_{8n+2}}{\lambda(\lambda - R_{8n+2}) - 1} \end{aligned} \tag{3.8}$$

By inserting the inequalities that we obtained for R_{8n+2} in (3.7) into (3.8) we obtain:

$$\begin{aligned} R_{8n+4} &= \frac{1}{\lambda - R_{8n+2}} \leq \frac{1}{\lambda - 1} \\ 1 - R_{8n+6} &= \frac{\lambda - 1 - R_{8n+2}}{\lambda(\lambda - R_{8n+2}) - 1} \leq \frac{\lambda - 2 + \sigma}{\lambda(\lambda - 1) - 1} \leq \sigma \end{aligned} \tag{3.9}$$

The very last inequality in (3.9) follows from the following observation:

$$\frac{\lambda - 2 + \sigma}{\lambda(\lambda - 1) - 1} \leq \sigma \Leftrightarrow \lambda - 2 + \sigma \leq \sigma(\lambda(\lambda - 1) - 1) \Leftrightarrow$$
$$(\lambda - 2)(\lambda - 1) + 1 \leq \lambda(\lambda - 1) - 1 \Leftrightarrow 4 \leq 2\lambda$$

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